Journal of Approximation Theory **92**, 379–410 (1998)
Article No. AT973131

Best Approximation of Functions like $|x|^{\lambda} \exp(-A\,|x|^{-\alpha})$

Michael I. Ganzburg*

Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, New York 10012
E-mail: ganzbrgm@acf4.nyu.edu

Communicated by Manfred v. Golitschek

Received January 16, 1996; accepted in revised form March 15, 1997

We determine the exact order of best approximation by polynomials and entire functions of exponential type of functions like $\varphi_{\lambda,\alpha}(x) = |x|^{\lambda} \exp(-A\,|x|^{-\alpha})$. In particular, it is shown that $E(\varphi_{\lambda,\alpha}, \mathscr{P}_n, L_p(-1,1)) \sim n^{-(2\lambda p + \alpha p + 2)/2p(1+\alpha)} \times \exp(-(1+\alpha^{-1})(A\alpha)^{1/(1+\alpha)} \cos \alpha\pi/2(1+\alpha)\, n^{\alpha/(1+\alpha)})$, where $E(\varphi_{\lambda,\alpha}, \mathscr{P}_n, L_p(-1,1))$ denotes best polynomial approximation of $\varphi_{\lambda,\alpha}$ in $L_p(-1,1)$, $\lambda \in \mathbb{R}$, $\alpha \in (0,2]$, $A > 0$, $1 \leqslant p \leqslant \infty$. The problem, concerning the exact order of decrease of $E(\varphi_{0,2}, \mathscr{P}_n, L_\infty(-1,1))$, has been posed by S. N. Bernstein.

1. INTRODUCTION

Let $E(f, B, F)$ denote best approximation of an element f from a normed space F by elements from a subspace $B \subset F$. Let $\mathscr{P}_n$ be the set of all algebraic polynomials of degree n or less and let B_σ be the set of all entire functions of exponential type σ.

This paper is devoted to a study of orders of best approximation of some individual infinitely differentiable functions by polynomials and entire functions of exponential type.

Similar problems, concerning polynomial, spline, rational, and harmonic approximation of individual functions of finite smoothness and analytic functions, have played an important role in Approximation Theory. For example, the well-known estimates of best polynomial approximation of $|x|$ given by Vallée-Poussin [20] and Bernstein [3] have initiated contemporary Approximation Theory.

Much attention has been attracted to the estimation of the degree of approximation of $|x|^{\alpha}$, by polynomials and entire functions of exponential type (Bernstein [4], Varga and Carpenter [21]), by rational functions

* This work was supported in part by the Program for Refugee Scientists in the New York Academy of Sciences.

0021-9045/98 $25.00

(Newman [13], Vjacheslavov [22], Stahl [17]), and by splines with variable knots (DeVore and Scherer [6]).

There has been extensive research on approximation of other individual functions (see [1, Appendices 44–46, 70, 86; 19, Chaps. 2, 7; 15, Chap. 4]). At the same time, no studies have been made of best approximation of individual infinitely differentiable functions.

In the late forties S. N. Bernstein posed a problem, concerning the exact order of decrease of $E(\varphi_{0,2}, \mathscr{P}_n, L_\infty(-1,1))$, where $\varphi_{\lambda,\alpha}(x) = |x|^\lambda \exp(-|x|^{-\alpha})$, $\lambda \in \mathbb{R}$, $\alpha > 0$ (this problem has been communicated to the author by Yu. A. Brudnyi.)

In the recent paper [10] we have obtained constructive descriptions of some classes of infinitely differentiable functions, including a generalized Gevrey class, and established the relations

$$\limsup_{n\to\infty} (E(\varphi_{0,1}, \mathscr{P}_n, L_\infty(-1,1)))^{n^{-1/2}} = \limsup_{\sigma\to\infty} (E(\varphi_{0,1}, B_\sigma, L_\infty(\mathbb{R})))^{\sigma^{-1/2}} = 0.243116.... \tag{1.1}$$

Some upper estimates of $E(\varphi_{\lambda,\alpha}, B_\sigma, L_p(\mathbb{R}))$, $1 \leqslant p \leqslant 2$, were obtained by the author and Liflyand [11].

In this paper we determine the exact order of best approximation by polynomials and entire functions of exponential type of the following functions

$$\varphi_{\lambda,\alpha,0}(x) = \begin{cases} x^\lambda \exp(-Ax^{-\alpha}), & x > 0, \\ 0, & x \leqslant 0, \end{cases}$$

$$\varphi_{\lambda,\alpha,i}(x) = (\operatorname{sgn} x)^{i+1} |x|^\lambda \exp(-A|x|^{-\alpha}), \qquad i = 1, 2,$$

$$\varphi_{\lambda,\alpha,3}(x) = \begin{cases} (a^2 - x^2)^\lambda \exp(-A((a+x)^{-\alpha} + (a-x)^{-\alpha})), & |x| < a, \\ 0, & |x| \geqslant a, \end{cases}$$

where $\alpha > 0$, $\lambda \in \mathbb{R}$, $a > 0$, $Re(A) > 0$.

In particular, we obtain the solution of the Bernstein problem

$$E(\varphi_{0,2}, \mathscr{P}_n, L_\infty(-1,1)) \sim n^{-1/3} \exp(-3\, 2^{-5/3} n^{2/3}).$$

A statement of main results is given in Section 2. Sections 5 and 6 contain the proofs of the upper and lower estimates of best approximation of $\varphi_{\lambda,\alpha,i}$. The proofs are based on new inequalities for best approximation of a function (Sect. 3) and on the asymptotics for Fourier coefficients and Fourier transforms of $\varphi_{\lambda,\alpha,i}$ (Sect. 4). Sections 7 and 8 contain applications of these results to the construction of fast decreasing polynomials and entire functions of exponential type (see Marchenko [1, p. 378], Nevai and Totik [12], Saff and Totik [16]).

Notation. Let $\mathbb{Z}$ be the set of all integers; $\mathbb{N}$ the set of all positive integers; $\mathbb{R}$ the real axis; $\mathbb{C}$ the complex plane; $\mathscr{P}_n$ the set of all algebraic polynomials with complex coefficients of degree n or less; and $\mathscr{T}_n$ the class of all trigonometric polynomials with complex coefficients of degree n or less.

Let us consider the class B_σ of all entire functions g of exponential type σ (i.e., such that for each $\varepsilon > 0$ there exists a constant C_ε satisfying the inequality $|g(z)| \leqslant C_\varepsilon \exp((\sigma+\varepsilon)|z|)$ for all $z \in \mathbb{C}$).

Let $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$, be the Banach space of measurable functions f on a measurable set $\Omega \subset \mathbb{R}$ with finite norm

$$\|f\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \, dx\right)^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_\Omega |f|, & p = \infty. \end{cases}$$

Let L_p^*, $1 \leqslant p \leqslant \infty$, be the Banach space of 2π-periodic measurable functions f with finite norm $\|f\|_{L_p^*} = \|f\|_{L_p(-\pi, \pi)}$, and let $C(\mathbb{R})$ be the normed space of the continuous functions f on $\mathbb{R}$ with finite norm $\|f\|_{C(\mathbb{R})} = \sup_{\mathbb{R}} |f|$.

Let us define for $\Omega \subset \mathbb{R}$, $1 \leqslant p \leqslant \infty$,

$$E(f, B, L_p(\Omega)) = \inf_{g \in B} \|f - g\|_{L_p(\Omega)},$$

where f is a measurable function on Ω and B is a linear set of measurable functions on Ω.

Let

$$\hat{f}(y) = \int_{\mathbb{R}} f(t) \exp(-iyt) \, dt, \qquad y \in \mathbb{R},$$

$$\hat{F}(k) = \int_{-\pi}^{\pi} F(t) \exp(-ikt) \, dt, \qquad k \in \mathbb{Z},$$

denote the Fourier transform and the Fourier coefficients of $f \in L_1(\mathbb{R})$ (or $f \in L_2(\mathbb{R})$) and $F \in L_1^*$, respectively.

Throughout, C will denote possibly different positive constants independent of k, n, σ, N, x, y, f, F.

For fixed numbers $\alpha > 0$, $\lambda \in \mathbb{R}$, $p \in [1, \infty)$, $A = |A| e^{i\theta}$, $|A| > 0$, $|\theta| < \pi/2$, we define the following constants

$$m_p = \frac{2\lambda p + \alpha p + 2}{2p(1+\alpha)}, \qquad m_\infty = \frac{\lambda + \alpha/2}{1+\alpha}; \tag{1.2}$$

$$b_p = \frac{2\lambda p - p + 2}{2p(1+\alpha)}, \qquad b_\infty = \frac{\lambda - 1/2}{1+\alpha}; \tag{1.3}$$

$$D_j = D_j(|A|, \theta)$$
$$= (1+\alpha^{-1})(|A|\,\alpha)^{1/(1+\alpha)} \exp\left(i \frac{\theta + (-1)^{j+1}\pi\alpha/2}{1+\alpha}\right), \qquad j = 1, 2, \tag{1.4}$$

$$M_\theta = \min_{j=1,2} Re(D_j)$$
$$= (1+\alpha^{-1})(|A|\,\alpha)^{1/(1+\alpha)} \cos((1+\alpha)^{-1} \max(|\theta + \pi\alpha/2|, |\theta - \pi\alpha/2|)).$$

It is easy to see that $M_\theta > 0$. In particular,

$$M_0 = (1+\alpha^{-1})(A\alpha)^{1/(1+\alpha)} \cos \frac{\alpha\pi}{2(1+\alpha)}.$$

We use the usual o, O notation, and also $\sim$ in the following sense: $F(\sigma) \sim G(\sigma)$ and $F(n) \sim G(n)$ if there exist constants $C_1, C_2 > 0$ independent of σ and n such that $C_1 \leqslant F(\sigma)/G(\sigma) \leqslant C_2$ for all $\sigma > 0$ and $C_1 \leqslant F(n)/G(n) \leqslant C_2$ for all $n \in \mathbb{N}$.

2. STATEMENT OF MAIN RESULTS

Following are our main estimates of best polynomial approximation of $\varphi_{\lambda,\alpha,i}$.

THEOREM 2.1. *For any $\lambda \in \mathbb{R}$, $\alpha \in (0, 2]$, $p \in [1, \infty]$, $Re(A) > 0$, $0 \leqslant i \leqslant 2$, $n \in \mathbb{N}$,*

$$E(\varphi_{\lambda,\alpha,i}, \mathscr{P}_n, L_p(-1,1)) \sim n^{-m_p} \exp(-M_\theta\, n^{\alpha/(1+\alpha)}). \tag{2.1}$$

As an immediate consequence of Theorem 2.1 for $\alpha = 2$, $p = \infty$, $\lambda = 0$, $A = 1$, $i = 1$, we obtain the solution of the Bernstein problem.

COROLLARY 2.1. *For $n \in \mathbb{N}$, $A = 1$,*

$$E(\varphi_{0,2,1}, \mathscr{P}_n, L_\infty(-1,1)) \sim n^{-1/3} \exp(-3\, 2^{-5/3}\, n^{2/3}).$$

Remark 2.1. The restriction $\alpha\leqslant 2$ is essential in Theorem 2.1. Using methods of this paper we can obtain the following relation for $\lambda\in\mathbb{R}$, $\alpha>0$, $A>0$, $1\leqslant p\leqslant\infty$, $0\leqslant i\leqslant 2$,

$$E(\varphi_{\lambda,\alpha,i},\mathscr{P}_n,L_p(-1,1))\sim n^{-m_p}\exp\left(-M_0\,n^{\alpha/(1+\alpha)}+\sum_{s=1}^{[\alpha/2]}a_s\,n^{(\alpha-2s)/(1+\alpha)}\right),$$

where a_s, $1\leqslant s\leqslant[\alpha/2]$, are real constants. We note that the problem of computation of these constants for any $\alpha>2$ appears to be very difficult.

A relation like (2.1) holds also for best approximation by entire functions of exponential type.

THEOREM 2.2. *For any* $\lambda\in\mathbb{R}$, $\alpha>0$, $p\in[1,\infty]$, $Re(A)>0$, $0\leqslant i\leqslant 3$, $\sigma>0$,

$$E(\varphi_{\lambda,\alpha,i},B_\sigma,L_p(\mathbb{R}))\sim\sigma^{-m_p}\exp(-M_\theta\,\sigma^{\alpha/(1+\alpha)}). \tag{2.2}$$

Finally, using the asymptotic behavior of the Fourier transform of $\varphi_{\lambda,\alpha,3}$, we obtain constructive proofs of the following results, concerning fast decreasing entire functions of exponential type and polynomials.

COROLLARY 2.2. (a) *For any* $\sigma>0$, $B>0$, $0<\beta<1$, *there exists a function* $g_\sigma\in B_\sigma$, $g_\sigma\not\equiv 0$, *such that*

$$|g_\sigma(y)|\leqslant C(\sigma|y|+1)^{-B}\exp(-M(\sigma|y|)^\beta),\qquad y\in\mathbb{R}, \tag{2.3}$$

where $C>0$ *and* $M>0$ *are constants independent of* σ *and* y.

(b) *If there exist* $C>0$, $M>0$, *and* $g_\sigma\in B_\sigma$ *such that* (2.3) *holds for* $\beta=1$, *then* $g_\sigma(y)\equiv 0$.

COROLLARY 2.3. (a) *For any* $B>0$, $0<\beta<1$, *there exists a sequence of polynomials* $P_n\in\mathscr{P}_n$ *such that* $P_n(0)=1$ *and for all* $y\in[-1,1]$

$$|P_n(y)|\leqslant C(n|y|+1)^{-B}\exp(-M(n|y|)^\beta),\qquad n\in\mathbb{N}, \tag{2.4}$$

where $C>0$ *and* $M>0$ *are constants independent of* n *and* y.

(b) *Inequality* (2.4) *is impossible for* $\beta=1$ *and any sequence of polynomials* $P_n\in\mathscr{P}_n$, $P_n(0)=1$, $n\in\mathbb{N}$.

Corollary 2.2(a) has been proved in [1, p. 378] and Corollary 2.3 has been obtained in [12].

3. ESTIMATES OF BEST APPROXIMATION

In [10] we obtained some estimates of best approximation of $\varphi_{0,1,1}$ using Jackson's and Bernstein's theorems for infinitely differentiable functions. This method makes it possible to find the exponential rate of best approximations like (1.1). To obtain relations like (2.1), (2.2) we need more precise inequalities. Here we establish lower and upper estimates of best approximation of a function f in the L_p-metric, $1 \leqslant p \leqslant \infty$, by polynomials and entire functions of exponential type, using Fourier coefficients and the Fourier transforms of f.

3.1. *Some Properties of Best Approximation*

We shall need some elementary properties of best approximation of periodic and nonperiodic functions.

The following lemma is easy to verify from the definition of $E(f, \mathscr{P}_n, L_p(-1, 1))$.

LEMMA 3.1. (a) *For any $f \in L_p(-1, 1)$, $1 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$,*

$$E(f, \mathscr{P}_n, L_p(-1,1)) = \begin{cases} 2^{-1/p} \inf\limits_{T \in \mathscr{T}_n} \left(\int_{-\pi}^{\pi} |f(\sin t) - T(t)|^p \, |\cos t| \, dt \right)^{1/p}, & 1 \leqslant p < \infty, \\ \inf\limits_{T \in \mathscr{T}_n} \operatorname*{ess\,sup}\limits_{[-\pi, \pi]} |f(\sin t) - T(t)|, & p = \infty. \end{cases} \tag{3.1}$$

(b) *If f is an even function from $L_p(-1, 1)$, $1 \leqslant p \leqslant \infty$, then for $n \in \mathbb{N}$*

$$E(f, \mathscr{P}_n, L_p(-1,1)) = \begin{cases} 2^{-1/p} \inf\limits_{T \in \mathscr{T}_{[n/2]}} \left(\int_{-\pi}^{\pi} |f(\sin(t/2)) - T(t)|^p \cos(t/2) \, dt \right)^{1/p}, & 1 \leqslant p < \infty, \\ \inf\limits_{T \in \mathscr{T}_{[n/2]}} \operatorname*{ess\,sup}\limits_{[-\pi, \pi]} |f(\sin(t/2)) - T(t)|, & p = \infty. \end{cases}$$

The following result is an immediate consequence of Lemma 3.1.

LEMMA 3.2. (a) *If $f \in L_p(-1, 1)$, $1 \leqslant p \leqslant \infty$, then for $F(t) = f(\sin t)$, $t \in [-\pi, \pi)$, $n \in \mathbb{N}$*

$$E(f, \mathscr{P}_n, L_p(-1,1)) \leqslant 2^{-1/p} E(F, \mathscr{T}_n, L_p^*).$$

(b) *If f is an even function from $L_p(-1,1)$, $1 \leqslant p \leqslant \infty$, then for* $F(t)=f(\sin(t/2))$, $t \in [-\pi, \pi)$, $n \in \mathbb{N}$

$$E(f, \mathscr{P}_n, L_p(-1,1)) \leqslant 2^{-1/p} E(F, \mathscr{T}_{[n/2]}, L_p^*).$$

LEMMA 3.3. (a) *If $f' \in L_p(-1,1)$, $1 \leqslant p \leqslant \infty$, then for $n \in \mathbb{N}$*

$$E(f, \mathscr{P}_n, L_p(-1,1)) \leqslant (C/n)\, E(f', \mathscr{P}_{n-1}, L_p(-1,1)). \tag{3.2}$$

(b) *Let f, f' be measurable functions on $\mathbb{R}$. If for some $\sigma > 0$, $E(f', B_\sigma, L_p(\mathbb{R})) < \infty$, $1 \leqslant p \leqslant \infty$, then*

$$E(f, B_\sigma, L_p(\mathbb{R})) \leqslant (C/\sigma)\, E(f', B_\sigma, L_p(\mathbb{R})). \tag{3.3}$$

Proof. A simple proof of (3.2) is given in [7, p. 220].

Inequality (3.3) follows from the estimate

$$E(F, B_\sigma, L_p(\mathbb{R})) \leqslant (\pi/(2\sigma))\, \|F'\|_{L_p(\mathbb{R})}, \qquad F' \in L_p(\mathbb{R}), \tag{3.4}$$

obtained by Krein (see [1, p. 244]). Indeed let $g_\sigma \in B_\sigma$ satisfy the equality

$$\|f' - g_\sigma\|_{L_p(\mathbb{R})} = E(f', B_\sigma, L_p(\mathbb{R})).$$

Putting $g_{\sigma 1}(x) = \int_0^x g_\sigma(t)\, dt$, we have $g_{\sigma 1} \in B_\sigma$ and $(f - g_{\sigma 1})' \in L_p(\mathbb{R})$. Thus using (3.4) for $F = f - g_{\sigma 1}$, we obtain

$$\begin{aligned} E(f, B_\sigma, L_p(\mathbb{R})) &= E(f - g_{\sigma 1}, B_\sigma, L_p(\mathbb{R})) \\ &\leqslant (\pi/(2\sigma))\, E(f', B_\sigma, L_p(\mathbb{R})). \end{aligned}$$

Hence (3.3) holds. ∎

3.2. *Upper Estimates*

Let $\Delta^2 a_k = a_k - 2a_{k+1} + a_{k+2}$ for any sequence $\{a_k\}_{k=0}^{\infty}$.

THEOREM 3.1. (a) *If $F(t) = a_0/2 + \sum_{k=1}^{\infty} a_k \cos kt$ in L_2^*, then for* $1 \leqslant p \leqslant 2$, $n \in \mathbb{N}$

$$\begin{aligned} E(F, \mathscr{T}_n, L_p^*) \leqslant C &\left(|a_{n+1}| + |a_{2n+2}| + \sum_{k=1}^{\infty} k\, |\Delta^2 a_{k+n}| \right)^{2/p-1} \\ &\times \left(\sum_{k=n+1}^{\infty} |a_k|^2 \right)^{1-1/p}. \end{aligned} \tag{3.5}$$

(b) *For any* $F \in L_p^*$, $q = p/(p-1)$, $2 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$

$$E(F, \mathcal{T}_n, L_p^*) \leqslant C\left(\sum_{|k| \geqslant n+1} |\hat{F}(k)|^q\right)^{1/q}. \tag{3.6}$$

Before giving the proof of the theorem we prove (3.5) for $p = 1$.

LEMMA 3.4. *If* $F(t) = a_0/2 + \sum_{k=1}^{\infty} a_k \cos kt$ *in* L_1^*, *then for the linear polynomial operator*

$$Q_n(F, t) = (a_0 - a_{2n+2})/2 + \sum_{k=1}^{n} (a_k - a_{2n+2-k}) \cos kt \tag{3.7}$$

the following inequalities hold

$$E(F, \mathcal{T}_n, L_1^*) \leqslant \|F - Q_n\|_{L_1^*}$$
$$\leqslant C\left(|a_{n+1}| + |a_{2n+2}| + \sum_{k=1}^{\infty} k\,|\Delta^2 a_{k+n}|\right). \tag{3.8}$$

Proof. First note that

$$F(t) - Q_n(F, t) = A_0/2 + \sum_{k=1}^{\infty} A_k \cos kt, \tag{3.9}$$

where

$$A_k = \begin{cases} a_k, & k \geqslant n+1, \\ a_{2n+2-k}, & 0 \leqslant k \leqslant n. \end{cases} \tag{3.10}$$

To estimate $\|F - Q_n\|_{L_1^*}$ we use the following inequality obtained by Bausov [2] (see also Telyakovskii [18])

$$\int_0^{\pi} \left| A_0/2 + \sum_{k=1}^{\infty} A_k \cos kt \right| dt$$
$$\leqslant C\left(|A_0| + |A_{n+1}| + \sum_{k=1}^{n+1} |A_{n+1-k} - A_{n+1+k}|/k\right.$$
$$\left. + \sum_{k=1}^{\infty} \frac{k\,|n+1-k|}{n+1+k}\,|\Delta^2 A_{k-1}|\right). \tag{3.11}$$

It follows from (3.10) that

$$A_0 = a_{2n+2}; \qquad A_{n+1} = a_{n+1}; \qquad A_{n+1-k} = a_{n+1+k}, \qquad 1 \leqslant k \leqslant n+1. \tag{3.12}$$

Hence $\sum_{k=1}^{n+1} |A_{n+1-k} - A_{n+1+k}|/k = 0$. It remains to estimate the second sum in the right-hand side of (3.11):

$$\sum_{k=1}^{\infty} \frac{k\,|n+1-k|}{n+1+k}\,|\Delta^2 A_{k-1}| = \sum_{k=1}^{n+1} + \sum_{k=n+2}^{\infty} = \mathrm{I}_1 + \mathrm{I}_2; \tag{3.13}$$

$$\mathrm{I}_1 \leqslant \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1}\,|\Delta^2 A_{k-1}| = \sum_{k=1}^{n} \frac{k(n+1-k)}{n+1}\,|\Delta^2 a_{n+k}|$$

$$\leqslant \sum_{k=1}^{\infty} k\,|\Delta^2 a_{k+n}|; \tag{3.14}$$

$$\mathrm{I}_2 = \sum_{k=n+2}^{\infty} \frac{k(k-n-1)}{n+k+1}\,|\Delta^2 a_{k-1}| \leqslant \sum_{k=n+2}^{\infty} (k-n-1)\,|\Delta^2 a_{k-1}|$$

$$= \sum_{k=1}^{\infty} k\,|\Delta^2 a_{k+n}|. \tag{3.15}$$

Combining relations (3.9)–(3.15), we obtain estimate (3.8). ∎

Proof of Theorem 3.1. First note that for any $F \in L_2^*$ and for the operator $Q_n(F, t)$, given by (3.7), we have

$$\|F - Q_n\|_{L_2^*} \leqslant 2E(F, \mathscr{T}_n, L_2^*). \tag{3.16}$$

Next, using Hölder's inequality for $s = (2-p)^{-1}$, $1 \leqslant p < 2$, we obtain

$$(E(F, \mathscr{T}_n, L_p^*))^p = \int_{-\pi}^{\pi} |F - Q_n|^p\,dt$$

$$\leqslant \left(\int_{-\pi}^{\pi} |F - Q_n|\,dt\right)^{1/s} \left(\int_{-\pi}^{\pi} |F - Q_n|^{(sp-1)/(p-1)}\,dt\right)^{p-1}$$

$$= \|F - Q_n\|_{L_1^*}^{2-p}\,\|F - Q_n\|_{L_2^*}^{2p-2}. \tag{3.17}$$

Now (3.5) follows from (3.8), (3.16), (3.17). Inequality (3.6) is an immediate consequence of Hausdorff-Young's theorem [23, p. 101] for $2 \leqslant p < \infty$, while (3.6) is evident for $p = \infty$, $q = 1$. ∎

The following result is an immediate consequence of Theorem 3.1 and Lemma 3.2.

COROLLARY 3.1. (a) *If* $f \in L_2(-1,1)$ *is an even function, then for* $1 \leqslant p < 2$, $n \in \mathbb{N}$, $N = [n/2]$

$$E(f, \mathscr{P}_n, L_p(-1,1)) \leqslant C\left(|a_{N+1}| + |a_{2N+2}| + \sum_{k=1}^{\infty} k\,|\Delta^2 a_{k+N}|\right)^{2/p-1} \left(\sum_{k=N+1}^{\infty} |a_k|^2\right)^{1-1/p}, \tag{3.18}$$

where $a_k = \int_{-\pi}^{\pi} f(\sin(t/2)) \cos kt\, dt$, $k \geqslant N+1$.

(b) *For any* $f \in L_p(-1,1)$, $q = p/(p-1)$, $2 \leqslant p \leqslant \infty$, $n \in \mathbb{N}$,

$$E(f, \mathscr{P}_n, L_p(-1,1)) \leqslant C\left(\sum_{|k| \geqslant n+1} |\hat{F}(k)|^q\right)^{1/q}, \tag{3.19}$$

where $F(t) = f(\sin t)$.

The following theorem is an integral analogue of Theorem 3.1.

THEOREM 3.2. (a) *Let* $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ *be an even continuous function on* $\mathbb{R}$ *and let* $\hat{f} \in L_1(\mathbb{R})$. *If* $\hat{f}(y)$ *and* $(d/dy)\hat{f}(y)$ *are locally absolutely continuous functions on* $[0,\infty)$ *such that* $\int_0^\infty y\,|(d^2/dy^2)\hat{f}(y)|\,dy < \infty$, *then for* $1 \leqslant p < 2$, $\sigma > 0$

$$E(f, B_\sigma, L_p(\mathbb{R})) \leqslant C\left(|\hat{f}(\sigma)| + |\hat{f}(2\sigma)| + \int_0^\infty y\left|\frac{d^2}{dy^2}\hat{f}(\sigma+y)\right| dy\right)^{2/p-1} \times \left(\int_\sigma^\infty |\hat{f}(y)|^2\,dy\right)^{1-1/p}. \tag{3.20}$$

(b) *If* $f \in L_1(\mathbb{R}) \cap C(\mathbb{R})$ *and* $\hat{f} \in L_1(\mathbb{R}) \cap L_q(\mathbb{R})$, *where* $q = p/(p-1)$, $2 \leqslant p \leqslant \infty$, *then for* $\sigma > 0$

$$E(f, B_\sigma, L_p(\mathbb{R})) \leqslant C\left(\int_{|y| \geqslant \sigma} |\hat{f}(y)|^q\,dy\right)^{1/q}. \tag{3.21}$$

Proof. Part (a) of the theorem has been proved in [11]. To establish (3.21) we note that $\hat{f} \in C(\mathbb{R})$. Hence the function

$$g(x) = (1/(2\pi)) \int_{-\sigma}^{\sigma} \hat{f}(y) \exp(ixy)\,dy$$

belongs to B_σ (cf. [23, p. 274]) and the following equality holds for all $x \in \mathbb{R}$

$$f(x) = (1/(2\pi)) \int_{\mathbb{R}} \hat{f}(y) \exp(ixy)\, dy. \tag{3.22}$$

Using the nonperiodic analogue of Hausdorff-Young's theorem [23, p. 254] and (3.22), we obtain for $2 \leqslant p < \infty$

$$E(f, B_\sigma, L_p(\mathbb{R})) \leqslant \|f - g\|_{L_p(\mathbb{R})} \leqslant C \|\hat{h}\|_{L_q(\mathbb{R})}, \tag{3.23}$$

where $h = f - g$. For $p = \infty$ (3.23) easily follows from (3.22). Taking account of the relation $\hat{h}(y) = 0$ for all $y \in (-\sigma, \sigma)$, we obtain (3.21) from (3.23). ▮

3.3. *Lower Estimates*

Let $\Delta_u^2 f(y) = f(y) - 2f(y+u) + f(y+2u)$ denote the second difference of f.

THEOREM 3.3. *If a function $f \in L_p(-1, 1)$, $1 \leqslant p \leqslant \infty$, satisfies the condition $x^{-2} f(x) \in L_1(-1, 1)$, then for each integer $m \geqslant n+1$*

$$E(f, \mathscr{P}_n, L_p(-1, 1)) \geqslant C \max_{j=1,2} \sup_{n \in \mathbb{N}} N^{1/q-2} |\Delta_N^2 F_j(m)|, \tag{3.24}$$

where $q = p/(p-1)$ and

$$F_j(y) = \int_{-\pi}^{\pi} f(\sin t) \sin^{-2} t \cos t\, (1 + \cos t) \exp((-1)^j ity)\, dt, \qquad j = 1, 2.$$

Proof. Using (3.1) and the weighted Hölder inequality, we obtain for any $N \in \mathbb{N}$, $m \geqslant n+1$, and $j = 1, 2$

$$\begin{aligned} &E(f, \mathscr{P}_n, L_p(-1, 1)) \\ &\quad = 2^{-1/p} \inf_{T_n \in \mathscr{T}_n} \left(\int_{-\pi}^{\pi} |f(\sin t) - T_n(t)|^p |\cos t|\, dt \right)^{1/p} \\ &\quad \geqslant 2^{-1/p} \left(\int_{-\pi}^{\pi} \left| \frac{\sin((N/2)\, t)}{\sin(t/2)} \right|^{2q} |\cos t|\, dt \right)^{-1/q} \\ &\qquad \times \inf_{T_n \in \mathscr{T}_n} \int_{-\pi}^{\pi} |f(\sin t) - T_n(t)| \left(\frac{\sin((N/2)\, t)}{\sin(t/2)} \right)^2 |\cos t|\, dt \end{aligned}$$

$$\geqslant C\, N^{1/q-2} \inf_{T_n \in \mathcal{T}_n} \left| \int_{-\pi}^{\pi} (f(\sin t) - T_n(t)) \right.$$

$$\left. \times \exp((-1)^j\, i(m+N)\, t) \left(\frac{\sin((N/2)\, t)}{\sin(t/2)} \right)^2 \cos t\, dt \right|$$

$$= C\, N^{1/q-2} \left| \int_{-\pi}^{\pi} f(\sin t) \cos t \exp((-1)^j\, i(m+N)\, t) \left(\frac{\sin((N/2)\, t)}{\sin(t/2)} \right)^2 dt \right|$$

$$= C\, N^{1/q-2} \left| \int_{-\pi}^{\pi} f(\sin t) \sin^{-2} t \cos t (1+\cos t)(\exp((-1)^j\, imt) \right.$$

$$\left. -2 \exp((-1)^j\, i(m+N)\, t) + \exp((-1)^j\, i(m+2N)\, t))\, dt \right|$$

$$= C\, N^{1/q-2}\, |\Delta_N^2 F_j(m)|,$$

and the theorem follows. ∎

Remark 3.1. The inequality

$$E(F^*, \mathcal{T}_n, L_p^*) \geqslant C \max_{j=1,2} \sup_{N \in \mathbb{N}} N^{1/q-2}\, |\Delta_N^2 F_j^*(m)|,$$

where $m \geqslant n+1$ and

$$F_j^*(y) = \int_{-\pi}^{\pi} F^*(t) \sin^{-2}(t/2) \exp((-1)^j\, ity)\, dt,$$

can be obtained similarly.

THEOREM 3.4. *If a function $f \in L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, satisfies the condition $x^{-2} f(x) \in L_1(\mathbb{R})$, then for each $\tau > \sigma > 0$*

$$E(f, B_\sigma, L_p(\mathbb{R})) \geqslant C \max_{j=1,2} \sup_{T>0} T^{1/q-2}\, |\Delta_T^2 F_j(\tau)|, \tag{3.25}$$

where $q = p/(p-1)$ and

$$F_j(y) = \int_{\mathbb{R}} x^{-2} f(x) \exp((-1)^j\, ixy)\, dx.$$

Proof. Putting $h_T(x) = (Tx)^{-2} \sin^2 Tx$, we remark that $g_{\sigma 1} = g_\sigma h_T$ belongs to $B_{\sigma+2T} \cap L_2(\mathbb{R})$ for any $g_\sigma \in B_\sigma \cap L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and each $T > 0$. Using the Paley–Wiener theorem [23, p. 274] we have

$\hat{g}_{\sigma 1}(y)=0$ for all y, $|y|>\sigma+2T$. Then using Hölder's inequality, we obtain for any $T>0$ and $\tau>\sigma>0$

$$
\begin{aligned}
&E(f, B_\sigma, L_p(\mathbb{R})) \\
&\quad \geqslant \|h_T\|^{-1}_{L_q(\mathbb{R})} \inf_{g_\sigma \in B_\sigma} \left| \int_{\mathbb{R}} (f-g_\sigma)(x) \exp((-1)^j i(\tau+2T)x) h_T(x)\, dx \right| \\
&\quad \geqslant CT^{1/q-2} \left| \int_{\mathbb{R}} x^{-2} f(x)(\exp((-1)^j i\tau x) - 2\exp((-1)^j i(\tau+2T)x) \right. \\
&\qquad \left. + \exp((-1)^j i(\tau+4T)x))\, dx \right|, \\
&\quad = CT^{1/q-2} |\Delta^2_{2T} F_j(\tau)|,
\end{aligned}
$$

giving (3.25). ∎

Remark 3.2. Note that different lower estimates for best approximation in the uniform metric have been obtained by Newman and Rivlin [14] and the author [9].

4. ASYMPTOTICS FOR FOURIER COEFFICIENTS AND FOURIER TRANSFORMS

Here we establish some asymptotic relations for Fourier coefficients and Fourier transforms of $\varphi_{\lambda, \alpha, i}$, $0 \leqslant i \leqslant 3$.

Some asymptotics for $\hat{\varphi}_{\lambda, \alpha, i}$ were obtained by Fedorjuk [8, p. 344] and by the author and Liflyand [11]. These results cannot be directly applied here since we consider a more general situation (in particular, we consider a complex A). Nevertheless, we shall apply the methods of [8, 11] with some changes.

Throughout this section $\mathbb{C}_+$ denotes the complex plane cut along the negative real axis and z^μ is the branch of this function in $\mathbb{C}_+$ which takes positive values for real $z>0$, $\mu>0$.

4.1. *Some Technical Asymptotics*

Let $A=|A| e^{i\theta}$, $|A|>0$, $|\theta|<\pi/2$. *For a fixed* $y>0$ *let us put*

$$
\begin{aligned}
&z_j = (|A|\alpha)^{1/(1+\alpha)} y^{-1/(1+\alpha)} \exp(i(\theta+(-1)^j \pi/2)/1+\alpha)), \\
&l_{Mj} = \{z \in \mathbb{C}_+ : z = \rho z_j, 0<\rho \leqslant M/|z_j|\}, \qquad 0<M\leqslant\infty, \quad j=1, 2.
\end{aligned}
\tag{4.1}
$$

LEMMA 4.1. *If* $\alpha>0,\ \lambda\in\mathbb{R},\ Re(A)>0,\ 0<M\leqslant\infty,\ j=1,2,\ y>0,$ *then for* $y\to\infty$

$$\int_{l_{Mj}} z^{\lambda}\exp(-Az^{-\alpha}+(-1)^{j}\,izy)\,dz$$
$$=(2\pi/(1+\alpha))^{1/2}\,(|A|\,\alpha)^{b_1}\exp(i(b_1\theta+(-1)^{j}\,\pi m_1/2))\,y^{-m_1}$$
$$\times\exp(-D_j\,y^{\alpha/(1+\alpha)})(1+O(y^{-\alpha/(1+\alpha)})),\tag{4.2}$$

where b_1, m_1, *and* D_j *are defined by* (1.2), (1.3), *and* (1.4), *respectively.*

Proof. We first note that the function $S_j(z,y)=-Az^{-\alpha}+(-1)^{j}\,izy$ satisfies the condition $(d/dz)\,S_j(z_j,y)=0$; that is, z_j is a simple saddle point of $S_j(z,y)$, $j=1,2$. The restriction $S_j(z,y)$ on l_{Mj} is the function in $\rho\geqslant 0$

$$S_j(\rho z_j,y)=(|A|\,\alpha)^{1/(1+\alpha)}\,y^{\alpha/(1+\alpha)}S_j(\rho),$$

where

$$S_j(\rho)=-\exp\left(i\,\frac{\theta+(-1)^{j+1}\,\pi\alpha/2}{1+\alpha}\right)(\rho^{-\alpha}/\alpha+\rho),\qquad j=1,2.$$

Next, we see that $\max_{\rho\geqslant 0} Re(S_j(\rho))=Re(Z_j(1))$, that is,

$$\max_{z\in l_{Mj}} Re(S_j(z,y))=Re(S_j(z_j,y)),$$

and z_j is the only extremal point of $Re(S_j(z,y))$ on l_{Mj}, $j=1,2$.

Applying now the Laplas method [5, Chap. 5; 8, Chaps. 2, 4, p. 263], we obtain

$$\int_{l_{Mj}} z^{\lambda}\exp(S_j(z,y))\,dz$$
$$=(|A|\,\alpha)^{(\lambda+1)/(1+\alpha)}\,y^{-(\lambda+1)/(1+\alpha)}\exp\left(i\,\frac{(2\lambda+1)(\theta+(-1)^{j}\,\pi/2)}{1+\alpha}\right)$$
$$\times\int_0^{M_1}\rho^{\lambda}\exp((|A|\,\alpha)^{1/(1+\alpha)}\,y^{\alpha/(1+\alpha)}S_j(\rho))\,d\rho$$
$$=(2\pi)^{1/2}\,(|A|\,\alpha)^{(\lambda+1)/(1+\alpha)}\,y^{-(\lambda+1)/(1+\alpha)}$$
$$\times\exp\left(i\,\frac{(2\lambda+1)(\theta+(-1)^{j}\,\pi/2)}{1+\alpha}\right)$$
$$\times(-\mu S_j''(1))^{-1/2}\exp(\mu S_j(1))(1+O(\mu^{-1})),\tag{4.3}$$

where $M_1 = M(|A| \alpha)^{-1/(1+\alpha)} y^{1/(1+\alpha)}$ and $\mu = (|A| \alpha)^{1/(1+\alpha)} y^{\alpha/(1+\alpha)}$. Thus (4.2) follows from (4.3). ▌

Let $\psi(z)$ be a function on $\mathbb{C}_+$ with the following asymptotic expansion in a neighborhood of the origin

$$\psi(z) = \sum_{\substack{0 \leqslant k\mu + s < n \\ o \leqslant k, s}} c_{k,s} z^{k\mu + s} + O(|z|^n) \tag{4.4}$$

for all $n \in \mathbb{N}$, where $\mu \geqslant 0$ is a fixed number.

LEMMA 4.2. *Let ψ be a bounded function on l_{Mj}, $0 < M < \infty$, $j = 1, 2$, and $\psi(0) = c_{0,0} \neq 0$. Then for $\alpha > 0$, $\lambda \in \mathbb{R}$, $Re(A) > 0$, $y > 0$, $y \to \infty$*

$$\begin{aligned}
&\int_{l_{Mj}} z^{\lambda} \psi(z) \exp(-Az^{-\alpha} + (-1)^j izy)\, dz \\
&\quad = \left(\frac{2\pi}{1+\alpha}\right)^{1/2} (|A| \alpha)^{b_1} \exp(i(b_1 \theta + (-1)^j \pi m_1/2))\, y^{-m_1} \\
&\qquad \times \exp(-D_j y^{\alpha/(1+\alpha)})(c_{0,0} + O(y^{-\tau})) + K(y),
\end{aligned} \tag{4.5}$$

where

$$\tau = \begin{cases} \min(\alpha, \mu, 1), & \text{if} \quad \mu > 0, \\ \min(\alpha, 1), & \text{if} \quad \mu = 0, \end{cases}$$

and $K(y)$ satisfies the inequality

$$|K(y)| \leqslant C y^{-m_1 - 1} \exp(-Re(D_j)\, y^{\alpha/(1+\alpha)}). \tag{4.6}$$

Proof. Using condition (4.4), we obtain

$$\begin{aligned}
&\int_{l_{Mj}} z^{\lambda} \psi(z) \exp(-Az^{-\alpha} + (-1)^j izy)\, dz \\
&\quad = \sum_{0 \leqslant k\mu + s < n} c_{k,s} \int_{l_{Mj}} z^{\lambda + k\mu + s} \exp(S_j(z, y))\, dz \\
&\qquad + \int_{l_{Mj}} R_n(z) \exp(S_j(z, y))\, dz \\
&\quad = I(y) + K(y),
\end{aligned} \tag{4.7}$$

where $n \geqslant 1+\alpha$ and $|R_n(z)| \leqslant C\,|z|^{n+\lambda}$ on l_{Mj}, $j=1, 2$. Then (4.5) follows from (4.7) and (4.2). It remains to estimate $K(y)$. Using the Laplas method (see [5, Chap. 5; 8, Chap. 2]) we obtain

$$\begin{aligned}
K(y) &\leqslant C \int_{l_{Mj}} |z|^{n+\lambda} \exp(Re(S_j(z, y)))\, dz \\
&\leqslant Cy^{-(\lambda+n+1)/(1+\alpha)} \int_0^\infty \rho^{n+\lambda} \exp\left(-(|A|\,\alpha)^{1/(1+\alpha)}\right. \\
&\quad \times \cos\left(\frac{\theta+(-1)^{j+1}\,\pi\alpha/2}{1+\alpha}\right) y^{\alpha/(1+\alpha)} \left.\left(\frac{\rho^\alpha}{\alpha}+\rho\right)\right) \\
&\leqslant Cy^{-(2\lambda+2n+2+\alpha)/2(1+\alpha)} \exp(-Re(D_j)\, y^{\alpha/(1+\alpha)}). \qquad (4.8)
\end{aligned}$$

Then (4.8) yields (4.6). ∎

4.2. *Asymptotics for Fourier Coefficients of* $\varphi_{\lambda, \alpha, i}(\sin t)$

LEMMA 4.3. *For* $\alpha \in (0, 2]$, $\lambda \in \mathbb{R}$, $Re(A) > 0$, $n \in \mathbb{N}$, $n \to \infty$, $j=1, 2$, *the following relations hold*

$$\begin{aligned}
I_{j,n} &= \int_{-\pi}^{\pi} \varphi_{\lambda,\alpha,0}(\sin t) \exp((-1)^j\, int)\, dt \\
&= \int_0^\pi \varphi_{\lambda,\alpha,1}(\sin t) \cos nt\, dt + (-1)^j\, i \int_0^\pi \varphi_{\lambda,\alpha,2}(\sin t) \sin nt\, dt \\
&= \int_0^\pi (\sin t)^\lambda \exp(-A(\sin t)^{-\alpha} + (-1)^j\, int)\, dt \\
&= C(-1)^{(j+1)n}\, n^{-m_1} (\exp(i(b_1\theta - \pi m_1/2)) \exp(-D_1 n^{\alpha/(1+\alpha)}) \\
&\quad + (-1)^n \exp(i(b_1\theta + \pi m_1/2)) \exp(-D_2 n^{\alpha/(1+\alpha)}))(1+O(n^{-\tau_1})) + K(n), \qquad (4.9)
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\pi/2} e^{2it} \varphi_{\lambda,\alpha,0}(\sin t) \exp((-1)^j\, 2int)\, dt \\
&\quad = Cn^{-m_1} \exp(i(b_1\theta + (-1)^j\, \pi m_1/2)) \\
&\qquad \times \exp(-D_j(2n)^{\alpha/(1+\alpha)})(1+O(n^{\tau_1})) + K(2n), \qquad (4.10)
\end{aligned}$$

$$\begin{aligned}
&\int_0^{\pi} \varphi_{\lambda,\alpha,0}(\sin t) \cos t(1+\cos t) \exp((-1)^j\, int)\, dt \\
&\quad = Cn^{-m_1} \exp(i(b_1\theta + (-1)^j\, \pi m_1/2)) \\
&\qquad \times \exp(-D_j n^{\alpha/(1+\alpha)})(1+O(n^{\tau_1})) + K(n), \qquad (4.11)
\end{aligned}$$

where $\tau_1 = (1+\alpha)^{-1} \min(\alpha, 2-\alpha)$ *and* K *is estimated by* (4.5).

Proof. First we prove (4.9) for $|\theta| < \pi/(2\alpha)$, $j=1$. Let

$$F_1 = \pi \sin\frac{\pi/2+\theta}{1+\alpha} \cos\frac{\pi/2-\theta}{1+\alpha} \Big/ \sin\frac{\pi}{1+\alpha},$$

$$F_2 = \pi \sin\frac{\pi/2+\theta}{1+\alpha} \sin\frac{\pi/2-\theta}{1+\alpha} \Big/ \sin\frac{\pi}{1+\alpha}.$$

Further, let

$$h_{\pm} = \{z \in \mathbb{C}: Re(z) = F_1,\ -F_2 < Im(z) < 0\}$$

be the vertical line segment oriented in a positive (or negative) sense. Let Γ_1 and Γ_2 denote the contours of the triangles with vertices $(0, 0)$, $(F_1, 0)$, $(F_1, -F_2)$ and $(F_1, 0)$, $(\pi, 0)$, $(F_1, -F_2)$, respectively, oriented in a negative sense.

Thus

$$\Gamma_1 = [0, F_1] \cup h_- \cup \gamma_1,$$

$$\Gamma_1 = [F_1, \pi] \cup \gamma_2 \cup h_+,$$

where γ_i denotes the hypotenuse of Γ_i, $i = 1, 2$.

Then we obtain

$$\int_0^{\pi} (\sin z)^{\lambda} \exp(-A(\sin z)^{-\alpha} - inz)$$

$$= \int_0^{F_1} + \int_{F_1}^{\pi} = \left(\int_{\Gamma_1} - \int_{h_-} - \int_{\gamma_1}\right) + \left(\int_{\Gamma_2} - \int_{h_+} - \int_{\gamma_1}\right)$$

$$= \int_{\Gamma_1} + \int_{\Gamma_2} - \int_{\gamma_1} - \int_{\gamma_2}. \tag{4.12}$$

Next, the function

$$H(z) = A((\sin z)^{-\alpha} - z^{-\alpha}) = Az^{2-\alpha} \left(\frac{\sin z}{z}\right)^{-\alpha} z^{-2} \left(1 - \left(\frac{\sin z}{z}\right)^{\alpha}\right)$$

is bounded in a neighborhood of the origin on $\mathbb{C}_+$ for $\alpha \in (0, 2]$. Hence in the sector $(\theta - \pi/2)/\alpha < \operatorname{Arg}(z) < (\theta + \pi/2)/\alpha$ we have

$$\lim_{z \to 0} (\sin z)^{\lambda} \exp(-A(\sin z)^{-\alpha} - inz)$$

$$= \lim_{z \to 0} (\sin z)^{\lambda} \exp(-H(z)) \exp(-Az^{-\alpha} - inz) = 0. \tag{4.13}$$

The equality $\int_{\Gamma_1}=0$ follows from (4.13). The relation $\int_{\Gamma_2}=0$ can be proved similarly. By (4.12) we obtain

$$\begin{aligned}\int_0^\pi &(\sin z)^\lambda \exp(-A(\sin z)^{-\alpha}-inz)\,dz\\ &=-\int_{\gamma_1}-\int_{\gamma_2}\\ &=\int_{l_{M1}} \psi(z)\exp(-Az^\alpha-inz)\,dz\\ &\quad+(-1)^n\int_{l_{M2}} \psi(z)\exp(-Az^{-\alpha}+inz)\,dz, \end{aligned}\tag{4.14}$$

where $M=\pi/(2\cos(\pi/(2(1+\alpha))))$, l_{Mj}, $j=1,2$, are defined by (4.1) and $\psi(z)=(\sin z/z)^\lambda\exp(H(z))$ is the bounded function on the l_{Mj}, $j=1,2$. It is easy to verify that ψ has asymptotic expansion (4.4) for $\mu=2-\alpha$, $c_{0,0}=-\alpha/3$. Then applying Lemma 4.2 and (4.14), we obtain (4.9) for $|\theta|<\pi/(2\alpha)$, $j=1$. Since $I_{2,n}=(-1)^n I_{1,n}$ (4.9) for $|\theta|<\pi/(2\alpha)$ follows.

Relations (4.9) for $|\theta|\geqslant\pi/(2\alpha)$, (4.10), and (4.11) can be proved similarly with minor changes. ∎

4.3. *Asymptotics for* $\hat{\varphi}_{\lambda,\alpha,3}$

LEMMA 4.4. *For any* $\alpha>0$, $\lambda\in\mathbb{R}$, $A>0$, $a>0$, $y>0$ $y\to\infty$,

$$\begin{aligned}\int_{\mathbb{R}} &\varphi_{\lambda,\alpha,3}(t)\exp(-ity)\,dt\\ &=\int_{-a}^{a}(a^2-z^2)^\lambda\exp(-A((z+a)^{-\alpha}+(a-z)^{-\alpha})-izy)\,dz\\ &=C\left(\exp\left(iay-i\pi\frac{2\lambda+2+\alpha}{4(1+\alpha)}\right)\exp(-D_1(A,0)\,y^{\alpha/(1+\alpha)})\right.\\ &\quad\left.+\exp\left(-iay+i\pi\frac{2\lambda+2+\alpha}{4(1+\alpha)}\right)\exp(-D_2(A,0)\,y^{\alpha/(1+\alpha)})\right)\\ &\quad\times(1+O(y^{-\tau_2}))+K(y), \end{aligned}\tag{4.15}$$

where $\tau_2=(1+\alpha)^{-1}\min(\alpha,1)$, *and* $K(y)$ *is estimated by* (4.5).

Proof. Let us define the saddle points $z_\pm=\pm a\mp(A\alpha)^{1/(1+\alpha)}y^{-1/(1+\alpha)}\times\exp(\mp i\pi/(2(1+\alpha)))$ and the line segments

$$l_{\pm a}=\{z\in\mathbb{C}\colon 0\leqslant|z\mp a|\leqslant\delta,\ \mathrm{Arg}(z\mp a)=\mp\pi/(2(1+\alpha))\},$$

where $\delta>0$ is a fixed number which can be chosen small enough. Let l_a^* denote the horizontal line segment joining l_{+a} and l_{-a}, so that

$$l_a^* = \{z\in\mathbb{C}: |Re(z)|<a(1-\cos(\pi/(2(1+\alpha)))) \quad \text{and} \quad Im(z)=\delta\sin(\pi/(2(1+\alpha)))\}.$$

Putting $\Gamma=[-a,a]\cup l_{+a}\cup l_a^*\cup l_{-a}$ oriented in a negative sense and denoting

$$f(z)=(a^2-z^2)^{\lambda}\exp(-A((z+a)^{-\alpha}+(a-z)^{-\alpha})-izy)$$

we obtain

$$\int_{-a}^{a} f(z)\,dz=\int_{\Gamma} f(z)\,dz-\int_{l_{-a}} f(z)\,dz-\int_{l_{+a}} f(z)\,dz-\int_{l_a^*} f(z)\,dz. \tag{4.16}$$

It is easy to verify that on the set

$$\{z\in\mathbb{C}: -\pi/(2\alpha)<\operatorname{Arg}(z+a)<0,\ -\pi/(2\alpha)<\operatorname{Arg}(a-z)<0\}$$

we have $\lim_{z\to\pm a} f(z)=0$. Hence

$$\int_{\Gamma} f(z)\,dz=0. \tag{4.17}$$

Furthermore,

$$\left|\int_{l_a^*} f(z)\,dz\right|\leqslant C\max_{z\in l_a^*}|f(z)|\leqslant C\exp\left(-\sin\frac{\pi}{2(1+\alpha)}\,\delta y\right). \tag{4.18}$$

Then

$$-\int_{l_{-a}} f(z)\,dz-\int_{l_a} f(z)\,dz=\exp(iay)\int_{l_{\delta 1}}\psi(z)\,z^{\lambda}\exp(-Az^{-\alpha}-izy)\,dz + \exp(-iay)\int_{l_{\delta 2}}\psi(z)\,z^{\lambda}\exp(-Az^{-\alpha}+izy)\,dz, \tag{4.19}$$

where $l_{\delta j}$, $j=1,2$, are defined by (4.1), and

$$\psi(z)=(2a-z)^{\lambda}\ \exp(-A(2a-z)^{-\alpha})$$

is an analytic function on $l_{\delta 1}\cup l_{\delta 2}$.

Applying now Lemma 4.2 for $\mu=0$, $c_{0,0}=\exp(-A(2a)^{-\alpha})(2a)^{\lambda}$, we obtain (4.15) from (4.16)–(4.19). ∎

4.4. *Asymptotics for* $\hat{\varphi}_{\lambda,\alpha,i}$, $0 \leqslant i \leqslant 2$

First we find the asymptotic behavior of $\hat{\varphi}_{\lambda,\alpha,i}$, $0 \leqslant i \leqslant 2$, for $\lambda < -1$ and then we extend this result for any $\lambda \in \mathbb{R}$.

LEMMA 4.5. *For any* $\alpha > 0$, $\lambda < -1$, $Re(A) > 0$, $j = 1, 2$, $y > 0$, $y \to \infty$,

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_{\lambda,\alpha,0}(t) \exp((-1)^j ity)\, dt \\
&\quad = \int_0^\infty \varphi_{\lambda,\alpha,1}(t) \cos ty\, dt + (-1)^j i \int_0^\infty \varphi_{\lambda,\alpha,2}(t) \sin ty\, dt \\
&\quad = \int_0^\infty z^\lambda \exp(-Az^{-\alpha} + (-1)^j izy)\, dz \\
&\quad = C \exp(i(b_1\theta + (-1)^j \pi m_1/2))\, y^{-m_1} \\
&\qquad \times \exp(-D_j y^{\alpha/(1+\alpha)})(1 + O(y^{-\alpha/(1+\alpha)})).
\end{aligned} \tag{4.20}$$

Proof. Let us define the contour

$$\Gamma_j = [0, R] \cup \Gamma_{Rj} \cup l_{Rj}, \qquad R > 0$$

oriented in a negative sense, where Γ_{Rj} are the arcs

$$\Gamma_{R1} = \left\{ z \in \mathbb{C} \colon |z| = R, \frac{\theta - \pi/2}{1+\alpha} < \operatorname{Arg}(z) < 0 \right\},$$

$$\Gamma_{R2} = \left\{ z \in \mathbb{C} \colon |z| = R, 0 < \operatorname{Arg}(z) < \frac{\theta + \pi/2}{1+\alpha} \right\},$$

and l_{Rj} are defined by (4.1), $j = 1, 2$.

The function $f(z) = z^\lambda \exp(-Az^{-\alpha} + (-1)^j izy)$ is analytic inside the curve Γ_j and $\lim_{R\to\infty} R \max_{z\in\Gamma_{Rj}} |f(z)| = 0$.

Hence

$$\int_{\Gamma_j} f\, dz = 0, \qquad \lim_{R\to\infty} \int_{\Gamma_{Rj}} f\, dz = 0. \tag{4.21}$$

Next, by (4.21)

$$\int_0^{\infty} f\,dz = \lim_{R \to \infty} \left(\left(\int_{\Gamma_j} f\,dz - \int_{\Gamma_{Rj}} f\,dz + \int_{l_{Rj}} \right) f\,dz = \int_{l_{\infty j}} f\,dz. \right) \tag{4.22}$$

Then (4.20) follows from (4.22) and Lemma 4.1. ∎

Since $\varphi_{\lambda, \alpha, 0}$ does not belong to $L_1(\mathbb{R})$ for $\lambda \geqslant -1$, formula (4.20) will be changed.

LEMMA 4.6. *For any $\alpha > 0$, $\lambda \geqslant -1$, $Re(A) > 0$, there exists an entire function g_0 of exponential type 1 such that $\|\varphi_{\lambda, \alpha, 0} - g_0\|_{L_1(\mathbb{R})} < \infty$, and for $y > 1$, $y \to \infty$, $j = 1, 2$,*

$$\begin{aligned} \int_{\mathbb{R}} (\varphi_{\lambda, \alpha, 0}(t) - g_0(t)) \exp((-1)^j ity)\,dt \\ = C \exp(i(b_1 \theta + (-1)^j \pi m_1/2))\, y^{-m_1} \\ \times \exp(-D_j(y^{\alpha/(1+\alpha)}))(1 + O(y^{\alpha/(1+\alpha)})). \end{aligned} \tag{4.23}$$

This asymptotic is an easy consequence of Lemma 4.5 and the following lemmas obtained in [11].

LEMMA 4.7. *Let $\lambda \in \mathbb{R}$ and $k = [|\lambda|] + 1$. Then there exists a function $g_0 \in B_1$ such that $\|\varphi_{\lambda, \alpha, 0} - g_0\|_{L_1(\mathbb{R})} < \infty$, and for $|y| > 1$, $m > k$*

$$\int_{\mathbb{R}} (\varphi_{\lambda, \alpha, 0}(t) - g_0(t)) \exp(-ity)\,dt = i^m y^{-m} \int_{\mathbb{R}} \varphi^{(m)}_{\lambda, \alpha, 0}(t) \exp(-ity)\,dt. \tag{4.24}$$

LEMMA 4.8. *For any $m \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$\varphi^{(m)}_{\lambda, \alpha, 0}(t) = \sum_{j=0}^{m} c_j \varphi_{\lambda - m - \alpha j, \alpha, 0}(t),$$

where c_j, $0 \leqslant j \leqslant m$, are some constants independent of t.

5. PROOF OF THE UPPER ESTIMATES OF THEOREMS 2.1 AND 2.2

The upper estimates of best approximation of $\varphi_{\lambda, \alpha, i}$ are based on Theorem 3.2, Corollary 3.1, Lemmas 4.3–4.8, and two auxiliary asymptotics [5, p. 14] given below:

$$\int_y^\infty t^{\beta-1}e^{-\mu t}\,dt=\mu^{-1}y^{\beta-1}e^{-\mu y}\left(1+\frac{\beta-1}{\mu y}+O(y^{-2})\right),\quad \mu>0,\quad \beta\in\mathbb{R},\quad y\to\infty, \tag{5.1}$$

$$\left(\int_\sigma^\infty t^{aq}\exp(-bqt^{\alpha/(1+\alpha)})\,dt\right)^{1/q} = C\sigma^{a+1/(q(1+\alpha))}\exp(-b\sigma^{\alpha/(1+\alpha)})(1+O(\sigma^{-\alpha/(1+\alpha)})),\quad a\in\mathbb{R},\quad b>0,\quad 1\leqslant q\leqslant\infty,\quad \sigma\to\infty. \tag{5.2}$$

5.1. *Approximation by Algebraic Polynomials*

Here we prove the upper estimates of Theorem 2.1.

First we assume that $2\leqslant p\leqslant\infty$. Using Corollary 3.1(b), (4.9), and (5.2), we obtain for $\lambda\in\mathbb{R}$, $0<\alpha\leqslant 2$, $0\leqslant i\leqslant 2$, $Re(A)>0$, $q=p/(p-1)$, $n\in\mathbb{N}$

$$\begin{aligned}E(\varphi_{\lambda,\alpha,i},\mathcal{P}_n,L_p(-1,1))&\leqslant C\left(\sum_{k=n+1}^\infty k^{-qm_1}\exp(-qM_\theta k^{\alpha/(1+\alpha)})\right)^{1/q}\\ &\leqslant C\left(\int_n^\infty y^{-qm_1}\exp(-qM_\theta y^{\alpha/(1+\alpha)})\,dy\right)^{1/q}\\ &\leqslant Cn^{-m_p}\exp(-M_\theta n^{\alpha/(1+\alpha)}).\end{aligned} \tag{5.3}$$

Let now $1\leqslant p<2$. We first prove the upper estimate of Theorem 2.1 for $\varphi_{\lambda,\alpha,1}$.

$\varphi_{\lambda,\alpha,1}$ is an even function, hence $\varphi_{\lambda,\alpha,1}(\sin(t/2))=\sum_{k=0}^\infty a_k\cos kt$, where the asymptotic for a_k is given by Lemma 4.3. Using (4.10), we obtain for all $k\in\mathbb{N}$

$$\begin{aligned}|\Delta^2 a_k|&=|a_k-2a_{k+1}+a_{k+2}|\\ &\leqslant C\max_{j=1,2}\left|\int_0^{\pi/2}e^{2it}\varphi_{\lambda+2,\alpha,0}(\sin t)\exp((-1)^j\,2ikt)\,dt\right|\\ &\leqslant Ck^{-m_1-2/(1+\alpha)}\exp(-M_\theta(2k)^{\alpha/(1+\alpha)}).\end{aligned} \tag{5.4}$$

Next, by (5.4) and (5.1),

$$\begin{aligned}\sum_{k=1}^\infty k\,|\Delta^2 a_{k+N}|&=\sum_{k=N+1}^\infty (k-N)\,|\Delta^2 a_k|\\ &\leqslant C\bigg(\int_N^\infty y^{-(m_1+2/(1+\alpha))+1}\exp(-M_\theta(2y)^{\alpha/(1+\alpha)}\,dy)\\ &\quad -n\int_{N+1}^\infty y^{-(m_1+2/(1+\alpha))}\exp(-M_\theta(2y)^{\alpha/(1+\alpha)})\,dy\bigg)\\ &\leqslant CN^{-m_1}\exp(-M_\theta(2N)^{\alpha/(1+\alpha)})\end{aligned} \tag{5.5}$$

for all $N \in \mathbb{N}$. Let us put $N=[n/2]$, $n \in \mathbb{N}$. Then taking account of the relations

$$|a_{2N+2}| \leqslant C\,|a_{N+1}| \leqslant CN^{-m_1} \exp(-M_\theta (2N)^{\alpha/(1+\alpha)}), \tag{5.6}$$

$$\left(\sum_{k=N+1}^{\infty} |a_k|^2\right)^{1/2} \leqslant C\left(\int_N^{\infty} y^{-2m_1} \exp(-2M_\theta (2y)^{\alpha/(1+\alpha)})\,dy\right)^{1/2}$$
$$\leqslant CN^{-m_2} \exp(-M_\theta (2N)^{\alpha/(1+\alpha)}), \tag{5.7}$$

we obtain from (3.18), (5.5), (5.6), (5.7)

$$E(\varphi_{\lambda,\alpha,1}, \mathscr{P}_n, L_p(-1,1))$$
$$\leqslant C\left(|a_{2N+2}| + |a_{N+1}| + \sum_{k=1}^{\infty} k\,|\Delta^2 a_{k+N}|\right)^{2/p-1} \left(\sum_{k=N+1}^{\infty} |a_k|^2\right)^{1-1/p}$$
$$\leqslant C^{N-((2/p-1)\,m_1 + (2-2/p)\,m_2)} \exp(-M_\theta (2N)^{\alpha/(1+\alpha)})$$
$$\leqslant Cn^{-m_p} \exp(-M_\theta n^{\alpha/(1+\alpha)}). \tag{5.8}$$

To establish the upper estimates of Theorem 2.1 for $\varphi_{\lambda,\alpha,i}$, $i=0,2$, $1 \leqslant p < 2$, we use the following relations

$$\varphi'_{\lambda,\alpha,2}(t) = \lambda \varphi_{\lambda-1,\alpha,1}(t) - \alpha A \varphi_{\lambda-1-\alpha,\alpha,1}(t), \tag{5.9}$$

$$\varphi_{\lambda,\alpha,0}(t) = (\varphi_{\lambda,\alpha,1}(t) + \varphi_{\lambda,\alpha,2}(t))/2. \tag{5.10}$$

Applying (3.2), (5.8), (5.9), and (5.10), we have for $1 \leqslant p < 2$

$$E(\varphi_{\lambda,\alpha,2}, \mathscr{P}_n, L_p(-1,1)) \leqslant (C/n)\, E(\varphi'_{\lambda,\alpha,2}, \mathscr{P}_n, L_p(-1,1))$$
$$\leqslant Cn^{-1} E(\varphi_{\lambda-1-\alpha,\alpha,1}, \mathscr{P}_n, L_p(-1,1))$$
$$\leqslant Cn^{-m_p} \exp(-M_\theta n^{\alpha/(1+\alpha)}), \tag{5.11}$$

$$E(\varphi_{\lambda\alpha,0}, \mathscr{P}_n, L_p(-1,1)) \leqslant Cn^{-m_p} \exp(-M_\theta n^{\alpha/(1+\alpha)}). \tag{5.12}$$

Relations (5.3), (5.8), (5.11), and (5.12) yield the upper estimates in (2.1). ▮

5.2. *Approximation by Entire Functions of Exponential Type*

We first prove the upper estimates of Theorem 2.2 for $2 \leqslant p \leqslant \infty$.

The functions $\varphi_{\lambda,\alpha,i}$, $0 \leqslant i \leqslant 2$, satisfy the conditions of Theorem 3.2(b) for $\lambda < -1$ (see (4.20)). Then, using (3.21), (4.20), (5.2), we obtain for $0 \leqslant i \leqslant 2$, $\lambda < -1$, $\alpha > 0$, $2 \leqslant p \leqslant \infty$, $q = p/(p-1)$

$$E(\varphi_{\lambda,\alpha,i}, B_\sigma, L_p(\mathbb{R})) \leqslant C\left(\int_\sigma^\infty y^{-qm_1}\exp(-qM_\theta y^{\alpha/(1+\alpha)})\,dy\right)^{1/q}$$
$$\leqslant C\sigma^{-m_p}\exp(-M_\theta \sigma^{\alpha/(1+\alpha)}), \tag{5.13}$$

giving the upper estimates in (2.2) for $\lambda < -1$, $2 \leqslant p \leqslant \infty$, $0 \leqslant i \leqslant 2$.

To prove the upper estimates of Theorem 2.2 for $\varphi_{\lambda,\alpha,i}$, $0 \leqslant i \leqslant 2$, $\lambda \geqslant -1$, $2 \leqslant p \leqslant \infty$, we shall use Lemma 4.6. There exists $g_0 \in B_1$ such that $\|\varphi_{\lambda,\alpha,0} - g_0\|_{L_1(\mathbb{R})} < \infty$. Then $g_i(t) = (g_0(t) + (-1)^{i+1} g_0(-t))/2$, $i = 1, 2$, belong to B_1 and $\|\varphi_{\lambda,\alpha,i} - g_i\|_{L_1(\mathbb{R})} < \infty$. Hence $\varphi_{\lambda,\alpha,i} - g_i$ satisfy the conditions of Theorem 3.2(b), $0 \leqslant i \leqslant 2$. By (3.21), (4.23), and (5.2), we obtain for $\sigma > 1$, $0 \leqslant i \leqslant 2$, $\lambda \geqslant -1$

$$E(\varphi_{\lambda,\alpha,i}, B_\sigma, L_p(\mathbb{R})) = E(\varphi_{\lambda,\alpha,i} - g_i, B_\sigma, L_p(\mathbb{R}))$$
$$\leqslant C\sigma^{-m_p}\exp(-M_\theta \sigma^{\alpha/(1+\alpha)}). \tag{5.14}$$

Then using (3.21), (4.15), and (5.2), we have for $\lambda \in \mathbb{R}$, $2 \leqslant p \leqslant \infty$

$$E(\varphi_{\lambda,\alpha,3}, B_\sigma, L_p(\mathbb{R})) \leqslant C\left(\int_\sigma^\infty y^{-qm_1}\exp(-qM_0 y^{\alpha/(1+\alpha)})\,dy\right)^{1/q}$$
$$\leqslant C\sigma^{-m_p}\exp(-M_0, \sigma^{\alpha/(1+\alpha)}). \tag{5.15}$$

Inequalities (5.13), (5.14), and (5.15) yield the upper estimates in Theorem 2.2 for $2 \leqslant p \leqslant \infty$.

These estimates are also valid for $1 \leqslant p < 2$. For a real A and the functions $\varphi_{\lambda,\alpha,i}$, $0 \leqslant i \leqslant 2$, these facts have been proved in [11]. Here we use the same approach with minor changes.

We first obtain the upper estimates for $\varphi_{\lambda,\alpha,i}$, $1 \leqslant p < 2$. Let $k = [|\lambda|] + 1$ and let $g_0 \in B_1$ satisfy (4.24). Putting

$$f(x) = \begin{cases} \varphi_{\lambda,\alpha,1}(x), & \lambda < -3, \\ \varphi_{\lambda,\alpha,1}(x) - (g_0(x) + g_0(-x))/2, & \lambda \geqslant -3, \end{cases}$$

we have for $\sigma > 1$, $1 \leqslant p < 2$

$$E(\varphi_{\lambda,\alpha,1}, B_\sigma, L_p(\mathbb{R})) = E(f, B_\sigma, L_p(\mathbb{R})). \tag{5.16}$$

Next, we prove the estimate

$$\left|\frac{d^l}{dy^l}\hat{f}(y)\right| \leqslant Cy^{-m_1 - l/(1+\alpha)}\exp(-M_\theta y^{\alpha/(1+\alpha)}) \tag{5.17}$$

for $\lambda \in \mathbb{R}$, $0 \leqslant l \leqslant 2$, $y > 1$.

If $\lambda < -3$, then, by Lemma 4.5,

$$\left|\frac{d^l}{dy^l}\hat{f}(y)\right| = \left|\frac{d^l}{dy^l}\hat{\varphi}_{\lambda,\alpha,1}(y)\right| \leqslant C\,|\hat{\varphi}_{\lambda+l,\alpha,0}(y)|$$
$$\leqslant Cy^{-m_1-l/(1+\alpha)}\exp(-M_\theta y^{\alpha/(1+\alpha)}). \tag{5.18}$$

If $\lambda \geqslant -3$, then, by Lemmas 4.7, 4.8, and 4.5,

$$\begin{aligned}\left|\frac{d^l}{dy^l}\hat{f}(y)\right| &\leqslant C\left|\frac{d^l}{dy^l}\widehat{(\varphi_{\lambda,\alpha,0}-g_0)}(y)\right| \\ &\leqslant C\left|\frac{d^l}{dy^l}\left(y^{-(k+3)}\int_{\mathbb{R}}\varphi^{(k+3)}_{\lambda,\alpha,0}(x)\exp(-ixy)\,dx\right)\right| \\ &= C(1+o(1))\,y^{-(k+3)}\left|\int_{\mathbb{R}}x^l\varphi_{\lambda,\alpha,0}(x)\exp(-ixy)\,dx\right| \\ &= C(1+o(1))\,y^{-(k+3)}\,|\hat{\varphi}_{\lambda+l-(k+3)(1+\alpha),\alpha,0}(y)| \\ &= C(1+o(1))\,y^{-m_1-l/(1+\alpha)}\exp(-M_\theta y^{\alpha/(1+\alpha)}), \qquad y\to\infty.\end{aligned} \tag{5.19}$$

Inequalities (5.18) and (5.19) imply (5.17) for $\lambda \in \mathbb{R}$, $0 \leqslant l < 2$, $y > 1$.
It follows from (5.17) and (5.1) that for $\sigma > 1$

$$\begin{aligned}\int_0^\infty y\left|\frac{d^2}{dy^2}\hat{f}(\sigma+y)\right|dy &= \int_\sigma^\infty (y-\sigma)\left|\frac{d^2}{dy^2}\hat{f}(y)\right|dy \\ &\leqslant C\left(\int_\sigma^\infty y^{-m_1-2/(1+\alpha)}\exp(-M_\theta y^{\alpha/(1+\alpha)})\,dy\right. \\ &\qquad \left.-\sigma\int_\sigma^\infty y^{-m_1-2/(1+\alpha)}\exp(-M_\theta y^{\alpha/(1+\alpha)})\,dy\right) \\ &\leqslant C\sigma^{-m_1}\exp(-M_\theta\sigma^{\alpha/(1+\alpha)}).\end{aligned} \tag{5.20}$$

Thus f satisfies the conditions of Theorem 3.2(a), and taking account of the estimates

$$|\hat{f}(2\sigma)| \leqslant C\,|\hat{f}(\sigma)| \leqslant C\sigma^{-m_1}\exp(-M_\theta\sigma^{\alpha/(1+\alpha)}), \tag{5.21}$$

$$\left(\int_\sigma^\infty |\hat{f}(y)|^2\,dy\right)^{1/2} \leqslant C\sigma^{-m_2}\exp(-M_\theta\sigma^{\alpha/(1+\alpha)}), \tag{5.22}$$

we obtain from (3.20), (5.16), (5.20), (5.21), and (5.22)

$$E(\varphi_{\lambda, \alpha, 1}, B_\sigma, L_p(\mathbb{R}))$$

$$\leqslant C\left(|\hat{f}(\sigma)| + |\hat{f}(2\sigma)| + \int_0^\infty y \left| \frac{d^2}{dy^2} \hat{f}(\sigma + y) \right| dy \right)^{2/p-1} \left(\int_\sigma^\infty |\hat{f}(y)|^2 \, dy \right)^{1-1/p}$$

$$\leqslant C\sigma^{-m_p} \exp(-M_\theta \sigma^{\alpha/(1+\alpha)}), \qquad 1 \leqslant p < 2. \tag{5.23}$$

The corresponding estimates for $\varphi_{\lambda, \alpha, i}$, $i = 0, 2$, $1 \leqslant p < 2$, follow from (3.3), (5.9), (5.10), and (5.23).

The upper estimate for $E(\varphi_{\lambda, \alpha, 3}, B_\sigma, L_p(\mathbb{R}))$, $1 \leqslant p < 2$, can be obtained similarly if we use Theorem 3.2(a) and Lemma 4.4.

The upper estimates of Theorem 2.2 are established. ∎

6. PROOFS OF THE LOWER ESTIMATES OF THEOREMS 2.1 AND 2.2

The lower estimates of best approximation of $\varphi_{\lambda, \alpha, i}$ are based on Theorems 3.3, 3.4, Lemmas 4.3–4.6, and some auxiliary results given below.

6.1. *Some technical Results*

LEMMA 6.1. *Let a and b be fixed real numbers, $b \neq 0$, and let $\psi(\sigma) = \cos(a + b\sigma^{\alpha/(1+\alpha)})$. Then for any $\sigma \geqslant |a/b|^{(1+\alpha)/\alpha}$ there exists $\tau > \sigma$ such that $|\psi(\tau)| = 1$ and*

$$\tau - \sigma < C\sigma^{1/(1+\alpha)}. \tag{6.1}$$

Proof. We may assume without loss of generality that $b > 0$. Putting

$$k = [(a + b\sigma^{\alpha/(1+\alpha)})/\pi] + 1, \qquad \tau = ((k\pi - a)/b)^{(1+\alpha)/\alpha}, \tag{6.2}$$

we have $\tau > \sigma$ and $|\psi(\tau)| = 1$. Furthermore,

$$\tau - \sigma < ((b\sigma^{\alpha/(1+\alpha)} + \pi)/b)^{(1+\alpha)/\alpha} - \sigma$$

$$< ((1+\alpha)\,\pi/(b\alpha))(\sigma^{\alpha/(1+\alpha)} + \pi/b)^{1/\alpha} < C\sigma^{1/(1+\alpha)}$$

giving (6.1). ∎

LEMMA 6.2. *Let $\psi(\sigma)$ be as in Lemma 6.1. Then for any $\sigma \geqslant |a/b|^{(1+\alpha)/\alpha}$ there exists $m \in \mathbb{N}$ such that $|\psi(m)| = 1 + o(1)$, $\sigma \to \infty$, and $0 < m - \sigma < C\sigma^{1/(1+\alpha)}$.*

Proof. Let τ be defined by (6.2). Putting $m = [\tau] + 1$ we have $m > \sigma$ and

$$|\psi(m)| = |\cos(b(m^{\alpha/(1+\alpha)} - \tau^{\alpha/(1+\alpha)}))| = 1 + O(\sigma^{-\alpha/(1+\alpha)}).$$

The inequality $m - \sigma < C\sigma^{1/(1+\alpha)}$ follows from (6.2). ∎

LEMMA 6.3. *Let a, b, and c be fixed real numbers, $c \neq 0$, and let $\psi(\sigma) = \cos(a + b\sigma^{\alpha/(1+\alpha)} + c\sigma)$. Then for any $\sigma > \sigma_0 = ((|a| + |b| + |c|)/|c|)^{1+\alpha}$ there exists $\tau > \sigma$ such that $|\psi(\tau)| = 1$ and $\tau - \sigma < C$.*

Proof. We may assume without loss of generality that $c > 0$. Then the function $h(\sigma) = a + b\sigma^{\alpha/(1+\alpha)} + c\sigma$ is monotone and positive for $\sigma > \sigma_0$. Putting $k = [h(\sigma)/\pi] + 1$ we define τ as the root of the equation $h(\tau) = k\pi$. Thus we have $\tau > \sigma$, $|\psi(\tau)| = 1$, and

$$\begin{aligned} \pi &\geqslant h(\tau) - h(\sigma) \geqslant C(\tau - \sigma) - |b|\,(\tau^{\alpha/(1+\alpha)} - \sigma^{\alpha/(1+\alpha)}) \\ &\geqslant (\tau - \sigma)\left(C - \frac{|b|\,\alpha}{(1+\alpha)\,\sigma^{1/(1+\alpha)}}\right). \end{aligned} \tag{6.3}$$

The inequality $\tau - \sigma < c$ follows from (6.3). ∎

LEMMA 6.4. *Let us put for fixed numbers $q \in [1, \infty]$, $B \in \mathbb{R}$, $D > 0$,*

$$I_{y,B,D} = \sup_{T>0} T^{1/q-2} |\Delta_T^2 F_{B,D}(y)|, \qquad y > 0,$$

where $\Delta_T^2 F(y) = F(y) - 2F(y+T) + F(y+2T)$ and

$$F_{B,D}(y) = y^B \exp(-Dy^{\alpha/(1+\alpha)})\,\psi(y) + o(y^B \exp(-Dy^{\alpha/(1+\alpha)})), \qquad y \to \infty. \tag{6.4}$$

Here $\psi(y)$ is one of the functions $\exp(i(a + by^{\alpha/(1+\alpha)}))$, $\cos(a + by^{\alpha/(1+\alpha)})$, $\cos(a + by^{\alpha/(1+\alpha)} + cy)$, where $a, b \neq 0$, $c \neq 0$ are fixed real numbers. Then there exists $\sigma_0 > 0$ such that for any $\sigma > \sigma_0$ there exists $\tau > 0$ satisfying the inequality

$$I_{\tau,B,D} \geqslant C\sigma^{B+(1-2q)/q(1+\alpha)} \exp(-D\sigma^{\alpha/(1+\alpha)}). \tag{6.5}$$

Proof. Putting $T_0 = C_0 y^{1/(1+\alpha)}$ for a fixed $y > 0$, we have

$$\begin{aligned} L_{y, B, D} &\geqslant T_0^{1/q-2} \, |\Delta_{T_0}^2 F_{B, D}(y)| \\ &\geqslant C y^{(1-2q)/q(1+\alpha)} (y^B \exp(-D y^{\alpha/(1+\alpha)}) \, |\psi(y)| \\ &\quad - 2(y + T_0)^B \exp(-D(y + T_0)^{\alpha/(1+\alpha)}) \\ &\quad - (y + 2T_0)^B \exp(-D(y + 2T_0)^{\alpha/(1+\alpha)})) \\ &\quad + o(y^B \exp(-D y^{\alpha/(1+\alpha)})), \qquad y \to \infty. \end{aligned} \tag{6.6}$$

Next, using the following inequalities

$$(y + C y^{1/(1+\alpha)})^B \leqslant y^B (1 + o(1)), \qquad y \to \infty,$$

$$(y + C_0 y^{1/(1+\alpha)})^{\alpha/(1+\alpha)} \geqslant y^{\alpha/(1+\alpha)} + \frac{C_0 \alpha}{1+\alpha} + o(1), \qquad y \to \infty,$$

we obtain from (6.6)

$$\begin{aligned} I_{y, B, D} &\geqslant C y^{(1-2q)/q(1+\alpha) + B} \exp(-D y^{\alpha/(1+\alpha)}) \\ &\quad \times \left(|\psi(y)| - 2 \exp\left(-D C_0 \frac{\alpha}{1+\alpha} + o(1) \right) \right. \\ &\quad \left. - \exp\left(-2 D C_0 \frac{\alpha}{1+\alpha} + o(1) \right) (1 + o(1)) + o(1) \right), \qquad y \to \infty. \end{aligned} \tag{6.7}$$

Now let C_0 and σ_0 be numbers satisfying the inequality ($y \to \infty$)

$$\begin{aligned} 1 - &\left(2 \exp\left(-D C_0 \frac{\alpha}{1+\alpha} + o(1) \right) \right. \\ &\left. + \exp\left(-2 D C_0 \frac{\alpha}{1+\alpha} + o(1) \right) \right) (1 + o(1)) + o(1) > 1/2 \end{aligned} \tag{6.8}$$

for all $y > \sigma_0$. If $\psi(y) = \exp(i(a + b y^{\alpha/(1+\alpha)}))$, then (6.5) follows from (6.7) and (6.8) for $\tau = \sigma + 1$. If $\psi(y) = \cos(a + b y^{\alpha/(1+\alpha)})$, $b \neq 0$, then Lemma 6.1, (6.7), and (6.8) yield (6.5). If $\psi(y) = \cos(a + b y^{\alpha/(1+\alpha)} + c y)$, $c \neq 0$, then (6.5) follows from Lemma 6.3, (6.7), and (6.8). ∎

LEMMA 6.5. *Let us put*

$$J_{n, B, D} = \sup_{N \in \mathbb{N}} N^{1/q-2} \, |\Delta_N^2 F_{B, D}(n)|,$$

where $F_{B,D}(y)$ *is defined by* (6.4) *and* $\psi(y)$ *is one of the functions* $\exp(i(a+by^{\alpha/(1+\alpha)}))$, $\cos(a+by^{\alpha/(1+\alpha)})$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for any* $n > n_0$ *there exists* $m \geqslant n+1$ *satisfying the inequalities*

$$J_{m,B,D} \geqslant Cn^{B+(1-2q)/q(1+\alpha)} \exp(-Dn^{\alpha/(1+\alpha)}).$$

The proof is similar to the proof of Lemma 6.4, if we apply Lemma 6.2 instead of Lemma 6.1.

6.2. *Proofs of the Lower Estimates*

It follows from (4.20) that

$$F_{j,s}(y) = \int_{\mathbb{R}} t^{-2} \varphi_{\lambda,\alpha,s}(t) \exp((-1)^j ity)\, dt, \qquad j=1,2; \quad 0 \leqslant s \leqslant 2,$$

coincides with $CF_{B,D}(y)$ given by (6.4), where

$$B = -m_1 + 2/(1+\alpha), \qquad D = Re(D_j) \tag{6.9}$$

and $\psi(y) = \exp(i(a+by^{\alpha/(1+\alpha)}))$ or $\psi(y) = \cos(a+by^{\alpha/(1+\alpha)})$. Here a and $b \neq 0$ are fixed real numbers independent of y. Next, using Theorem 3.4 and Lemma 6.4, we obtain for $\lambda \in \mathbb{R}$, $\alpha > 0$, $q = p/(p-1)$, $1 \leqslant p \leqslant \infty$, $0 \leqslant s \leqslant 2$, $\tau > \sigma > \sigma_0$

$$\begin{aligned} E(\varphi_{\lambda,\alpha,s}, B_\sigma, L_p(\mathbb{R})) &\geqslant C \max_{j=1,2} I_{\tau,\, -m_1+2/(1+\alpha),\, Re(D_j)} \\ &\geqslant C \max_{j=1,2} \sigma^{-m_1+(q(1+\alpha))^{-1}} \exp(-Re(D_j)\, \sigma^{\alpha/(1+\alpha)}) \\ &= C\sigma^{-m_p} \exp(-M_\theta \sigma^{\alpha/(1+\alpha)}). \end{aligned} \tag{6.10}$$

Then Lemma 4.4 shows that $F_{j,3}(y) = CF_{B,D}(y)$ where B, D are defined by (6.9) and $\psi(y) = \cos(a+by^{\alpha/(1+\alpha)}+cy)$, $c \neq 0$. Using Theorem 3.4 and Lemma 6.4 we obtain (6.10) for $s=3$ as well.

Furthermore, (4.11) implies the relations for $j=1,2$, $n \in \mathbb{N}$

$$\begin{aligned} &\int_{-\pi}^{\pi} \varphi_{\lambda,\alpha,s}(\sin t) \sin^{-2} t \cos t(1+\cos t) \exp((-1)^j int)\, dt \\ &= \begin{cases} \int_0^{\pi} \varphi_{\lambda-2,\alpha,s}(\sin t) \cos t(1+\cos t) \exp((-1)^j int)\, dt, & s=0, \\ \int_0^{\pi} \varphi_{\lambda-2,\alpha,s}(\sin t) \cos t(1+\cos t)(\exp((-1)^j int)) & \\ \qquad +(-1)^{s+1} \exp((-1)^{j+1} int)\, dt, & s=1,2, \end{cases} \\ &= \begin{cases} C_j F_{B,Re(D_j)}(n), & s=0 \\ C_j F_{B,M_\theta}(n), & s=1,2, \end{cases} \end{aligned}$$

where $B=-m_1+2/(1+\alpha)$, $\psi(y)=\exp(i(a+by^{\alpha/(1+\alpha)}))$ or $\psi(y)=\cos(a+by^{\alpha/(1+\alpha)})$ and $C_j\neq 0$, $j=1, 2$ are constants independent of y. Using Theorem 3.3 and Lemma 6.5, we obtain that there exists $m\in\mathbb{N}$, $m\geqslant n+1$, such that

$$E(\varphi_{\lambda,\alpha,s}, \mathscr{P}_n, L_p(-1,1))\geqslant CJ_{m,\,-m_1+2/(1+\alpha),\,M_\theta}\geqslant Cn^{-m_p}\exp(-M_\theta n^{\alpha/(1+\alpha)}) \tag{6.11}$$

for $0\leqslant s\leqslant 2$, $0<\alpha\leqslant 2$, $\lambda\in\mathbb{R}$, $1\leqslant p\leqslant\infty$.

The lower estimates of Theorem 2.1 and 2.2 follow from (6.10), (6.11). ∎

7. PROOF OF COROLLARY 2.2

Putting $a=1$, $\alpha=\beta/(1-\beta)$, $\lambda=(2B+\beta-2)/2(1-\beta)$,

$$g_\sigma(y)=\int_{-1}^{1}\varphi_{\lambda,\alpha,3}(t)\exp(i\sigma ty)\,dt, \qquad y\in\mathbb{R}, \tag{7.1}$$

we have $g_\sigma\in B_\sigma$. If $\sigma\,|y|>1$, then (2.3) follows from Lemma 4.4. If $0\leqslant\sigma\,|y|\leqslant 1$, then for any $M>0$

$$\begin{aligned}|g_\sigma(y)|&\leqslant\int_{-1}^{1}\varphi_{\lambda,\alpha,3}(t)\,dt\\&\leqslant 2^Be^M\int_{-1}^{1}\varphi_{\lambda,\alpha,3}(t)\,dt(\sigma\,|y|+1)^{-B}\exp(-M(\sigma\,|y|)^\beta).\end{aligned}$$

Hence (2.3).

Let now $g\in B_\sigma$ satisfy (2.3) for $\beta=1$, $C>0$, $M>0$. Then $g\in L_2(\mathbb{R})$ and by the Paley–Wiener theorem, $\hat{g}(y)=0$ for $|y|>\sigma$. Next, for any $\tau>0$

$$E(\hat{g}, B_\tau, L_\infty(\mathbb{R}))\leqslant\sup_{y\in\mathbb{R}}\left|\hat{g}(y)-\int_{-\tau}^{\tau}g(t)\,e^{-ity}\,dt\right|\leqslant\int_{|t|\geqslant\tau}|g(t)|\,dt\leqslant Ce^{-M\tau}.$$

Hence, by Bernstein's converse theorem [19, p. 370] $\hat{g}$ is an analytic function on $\mathbb{R}$. This implies $\hat{g}(y)=0$ for all $y\in\mathbb{R}$. Corollary 2.2(b) follows. ∎

8. PROOFS OF COROLLARY 2.3

Let us put $\psi_\sigma = C_0 g_\sigma$, where g_σ is defined by (7.1) and $C_0 = (g_\sigma(0))^{-1}$.

To prove Corollary 2.3(a) we need the following result obtained by Bernstein [19, p. 283]: for any $\gamma \in (0, 1)$, $\sigma > 0$,

$$E(\psi_\sigma, \mathscr{P}_n, L_\infty(-\gamma n/\sigma, \gamma n/\sigma)) = \max_{|y| \leqslant 1} |\psi_\sigma(\gamma n y/\sigma) - Q_n(\gamma n y/\sigma)| \leqslant C e^{-Hn}, \tag{8.1}$$

where Q_n is the polynomial of best approximation of ψ_σ in the metric $L_\infty(-\gamma n/\sigma, \gamma n/\sigma)$ and C, H are positive constants depending only on γ.

It follows from (8.1) that $Q_n(0) = 1 + o(1)$, $n \to \infty$. Hence there exists $n_0 \in \mathbb{N}$ such that $Q_n(0) > 0$, $n > n_0$. Putting $P_n(y) = Q_n(\gamma n y/\sigma)/Q_n(0)$, $n > n_0$, we obtain from (8.1) and (2.3)

$$\begin{aligned} |P_n(y)| &\leqslant (|\psi_\sigma(\gamma n y/\sigma)| + |\psi_\sigma(\gamma n y/\sigma) - Q_n(\gamma n y/\sigma)|)(1 + o(1)) \\ &\leqslant C(n|y| + 1)^{-B} \exp(-M(n|y|)^{\beta}) + C \exp(-Hn) \\ &\leqslant C(n|y| + 1)^{-B} \exp(-M(n|y|)^{\beta}), \qquad |y| \leqslant 1, \end{aligned} \tag{8.2}$$

Since (2.4) is valid for arbitrary polynomials $P_n \in \mathscr{P}_n$, $1 \leqslant n \leqslant n_0$, Corollary 2.3(a) follows from (8.2).

To prove Corollary 2.3(b) we assume that there exists a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$, $P_n(0) = 1$, $n \in \mathbb{N}$, satisfying (2.4) for $\beta = 1$. Setting $Q_n(y) = P_n(y/n) = 1 + \sum_{k=1}^{n} a_{kn} y^k$ we have from (2.4) for $\beta = 1$

$$|Q_n(y)| \leqslant C e^{-M|y|} \leqslant C, \qquad y \in [-n, n].$$

By V. V. Markov's inequality [19, p. 227], $|a_{kn}| \leqslant C/k!$, $1 \leqslant k \leqslant n$. Hence there exists a subsequence $\{Q_{n_s}\}_{s=1}^{\infty}$ such that Q_{n_s} converges uniformly on each finite part of $\mathbb{R}$ to a certain function $g_1 \in B_1$ as $s \to \infty$ (see [19, p. 50]). Next, $g_1(0) = 1$ and $|g_1(y)| \leqslant C \exp(-M|y|)$ for any $y \in \mathbb{R}$. Now Corollary 2.3(b) follows from Corollary 2.2(b). ▮

ACKNOWLEDGMENT

The author would like to thank Professor Yu. A. Brudnyi for information regarding the Bernstein problem and many inspiring discussions.

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